ON SOME PERTURBATIONS OF A SYMMETRIC STABLE PROCESS AND THE CORRESPONDING CAUCHY PROBLEMS

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Abstract. A semigroup of linear operators on the space of all continuous bounded functions given on a d-dimensional Euclidean space $\mathbb{R}^d$ is constructed such that its generator can be written in the following form $\mathbf{A}+(a(\cdot),\mathbf{B})$, where $\mathbf{A}$ is the generator of a symmetric stable process in $\mathbb{R}^d$ with the exponent $\alpha\in(1,2]$, $\mathbf{B}$ is the operator that is determined by the equality $\mathbf{A}=c\,\mathbf{div}(\mathbf{B})$ ($c>0$ is a given parameter), and a given $\mathbb{R}^d$-valued function $a\in L_p(\mathbb{R}^d)$ for some $p>d+\alpha$ (the case of $p=+\infty$ is not exclusion). However, there is no Markov process in $\mathbb{R}^d$ corresponding to this semigroup because it does not preserve the property of a function to take on only non-negative values. We construct a solution of the Cauchy problem for the parabolic equation $\frac{\partial u}{\partial t}=(\mathbf{A}+(a(\cdot),\mathbf{B}))u$.

Introduction

A d-dimensional symmetric stable process (α-stable process) is a Markov process in $\mathbb{R}^d$ with its transition probability density given by

$$g(t,x,y)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\exp\{i(y-x,\xi)-c\,t|\xi|^\alpha\}\,d\xi,\quad t>0,\ x\in\mathbb{R}^d,\ y\in\mathbb{R}^d$$

(parameters $c>0$ and $\alpha\in(1,2]$ will be fixed throughout this article). As is well known, the generator $\mathbf{A}$ of this process is a pseudo-differential operator, whose symbol is given by the expression $(-c|\lambda|^\alpha)_{\lambda\in\mathbb{R}^d}$.The parameter α is called the exponent of this process.

A Wiener process is a particular case of a symmetric stable process, if we put $\alpha=2$ and $c=1/2$. Its generator is the Laplace operator (with the multiplier $1/2$). The perturbation of this operator by means of the operator (a,∇), where $(a(x))_{x\in\mathbb{R}^d}$ is some $\mathbb{R}^d$-valued function, ∇ is the Hamilton operator (gradient) and $(\cdot,\cdot)$ denotes the scalar product in $\mathbb{R}^d$, allows us to construct the diffusion process with the drift vector a. A great deal of publications considered perturbations under some more or less general assumptions on the function a (see, for example, [5] and the references therein).

This article is devoted to the perturbing a symmetric stable process with $\alpha\in(1,2)$ in a similar way. In our situation the operator $\mathbf{B}$, with its symbol $(i|\lambda|^{\alpha-2}\lambda)_{\lambda\in\mathbb{R}^d}$, is an analogue to the gradient. The role of this operator in the theory of potentials for symmetric stable processes is discussed in the paper [9].

Symmetric stable processes were perturbed by terms of the type (a,∇) under various assumptions on the function a in many papers (see, for example, [2, 4, 10, 11]). The perturbation of stable processes with delta-function in coefficient is constructed in [6, 8]. The operator $\mathbf{B}$ used in perturbations of stable processes in the papers [6, 7, 8].

This paper is organized as follows. In the next section we present the basic concepts and preliminary results. Section 2 contains the construction of the stable process perturbation and the investigation of some its properties. And the final Section 3 is devoted to the Cauchy problem for the pseudo-differential equation of parabolic type with operator $\mathbf{A}+(a,\mathbf{B})$ on the spatial variable.

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1. Notation and auxiliary results

Let F_γ ($\gamma > 0$) be the class of functions $\varphi(x)$ defined on $\mathbb{R}^d$ with values in $\mathbb{R}$, which are the Fourier transforms $\varphi(x) = \int_{\mathbb{R}^d} e^{i(x,\lambda)}\Phi(\lambda)\,d\lambda$ and such that the functions $|\lambda|^\gamma\Phi(\lambda)$ are absolutely integrable on $\mathbb{R}^d$.

Recall that the operator $\mathbf{A}$ acting on the functions $\varphi \in F_\alpha$ according to the following rule $\mathbf{A}\varphi(x) = -c\int_{\mathbb{R}^d} |\lambda|^\alpha e^{i(x,\lambda)}\Phi(\lambda)\,d\lambda$ and the equality $\mathbf{B}\varphi(x) = \int_{\mathbb{R}^d} i|\lambda|^{\alpha-2}\lambda e^{i(x,\lambda)}\Phi(\lambda)\,d\lambda$ is true for functions $\varphi \in F_{\alpha-1}$. It is easy to see that the equality $\mathbf{A} = c\,\mathbf{div}(\mathbf{B})$ holds on $F_{\alpha-1}$, where $\mathbf{div}$ is the divergence operator.

Let $(a(x))_{x\in\mathbb{R}^d}$ be a some given $\mathbb{R}^d$-valued measurable function.

Definition 1.1. A function $(G(t,x,y))_{t>0,x\in\mathbb{R}^d,y\in\mathbb{R}^d}$ is called a result of perturbing the transition probability density $g(t,x,y)$, if it is a solution of the following equation

$$G(t,x,y) = g(t,x,y) + \int_0^t d\tau \int_{\mathbb{R}^d} g(t-\tau,x,z)(a(z),\mathbf{B}_z G(\tau,z,y))\,dz. \tag{1}$$

The subscript of operator $\mathbf{B}$ (here and in what follows) means that it acts on a function of several variables in the indicated variable.

We will construct the solution of equality (1) in the form

$$G(t,x,y) = g(t,x,y) + \int_0^t d\tau \int_{\mathbb{R}^d} g(t-\tau,x,z)V(\tau,z,y)|a(z)|\,dz, \tag{2}$$

where the function $V(t,x,y)$ satisfies the equation

$$V(t,x,y) = V_0(t,x,y) + \int_0^t d\tau \int_{\mathbb{R}^d} V_0(t-\tau,x,z)V(\tau,z,y)|a(z)|\,dz \tag{3}$$

and

$$V_0(t,x,y) = (\mathbf{B}_x g(t,x,y), e(x)) = \frac{1}{c\,\alpha}\frac{(y-x,e(x))}{t}g(t,x,y). \tag{4}$$

Here we use a function $(e(x))_{x\in\mathbb{R}^d}$ defined by the equality $e(x) = \frac{1}{|a(x)|}a(x)$ for $x \in \mathbb{R}^d$ such that $|a(x)| \neq 0$ and an arbitrary value (with preservation of the measurability) otherwise.

Equation (3) can be solved by the method of successive approximations, namely its solution will be found in the form

$$V(t,x,y) = \sum_{k=0}^{\infty} V_k(t,x,y), \tag{5}$$

where $V_0(t,x,y)$ is defined by equality (4) and for $k \geq 1$ the following equality

$$V_k(t,x,y) = \int_0^t d\tau \int_{\mathbb{R}^d} V_0(t-\tau,x,z)V_{k-1}(\tau,z,y)|a(z)|\,dz$$

is valid.

We will use some inequalities that are proved in the article [3].

The first inequality is

$$g(t,x,y) \leq N\frac{t}{(t^{1/\alpha}+|y-x|)^{d+\alpha}}, \tag{6}$$

where $N > 0$ is a constant, $t > 0$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}^d$.

The following inequality will be used in various situations

$$
\begin{aligned}
&\int_0^t d\tau \int_{\mathbb{R}^d} \frac{(t-\tau)^{\beta/\alpha}}{((t-\tau)^{1/\alpha}+|z-x|)^{d+\alpha+k}} \frac{\tau^{\gamma/\alpha}}{(\tau^{1/\alpha}+|y-z|)^{d+\alpha+l}}\, dz \le \\
(7)\qquad &\le C\left[B\left(\frac{\beta-k}{\alpha}, 1+\frac{\gamma}{\alpha}\right) t^{\frac{\beta+\gamma-k}{\alpha}} \frac{1}{(t^{1/\alpha}+|y-x|)^{d+\alpha+l}} + \right.\\
&\left. + B\left(1+\frac{\beta}{\alpha}, \frac{\gamma-l}{\alpha}\right) t^{\frac{\beta+\gamma-l}{\alpha}} \frac{1}{(t^{1/\alpha}+|y-x|)^{d+\alpha+k}}\right],
\end{aligned}
$$

that is true for some constants β, γ, k, l, satisfying the conditions: $-\alpha < k < \beta$, $-\alpha < l < \gamma$, and $C > 0$ which depends only on d, α, k and l. Here $B(\cdot,\cdot)$ is Euler beta function.

We shall also use below the following result (see, for example, [3]). Denote by $C_b(D)$ the space of all continuous bounded real-valued functions on the set D. Let $\varphi \in C_b(\mathbb{R}^d)$ and $(f(t,x))_{t\ge 0, x\in\mathbb{R}^d}$ be a continuous function bounded on each domain of the form $D_T = [0,T]\times\mathbb{R}^d$ for $T < +\infty$. We suppose that the function f is Hölder continuous (with an arbitrary coefficient from the interval $(0,1)$) in the argument x locally uniformly with respect to t. Then the unique bounded solution of the Cauchy problem

$$
(8)\qquad \begin{cases} \frac{\partial u(t,x)}{\partial t} = \mathbf{A}_x u(t,x) + f(t,x), & t>0,\ x\in\mathbb{R}^d, \\ \lim\limits_{t\to 0+} u(t,x) = \varphi(x), & x\in\mathbb{R}^d \end{cases}
$$

can be written as follows

$$
u(t,x) = \int_{\mathbb{R}^d} g(t,x,y)\varphi(y)\, dy + \int_0^t d\tau \int_{\mathbb{R}^d} g(t-\tau,x,z) f(\tau,z)\, dz.
$$

2. The perturbation

In this section we will prove existence of the perturbation (in the sense of Definition 1.1) by operator $\mathbf{B}$ with the function a satisfies some integrability condition. A few properties of this perturbation will be established below.

Theorem 2.1. *Let the function $(a(x))_{x\in\mathbb{R}^d}$ satisfies the following condition: $a\in L_p(\mathbb{R}^d)$ with $p > d+\alpha$ (maybe, $p=+\infty$).*

Then the perturbation $G(t,x,y)$ (see Definition 1.1) exists and possesses the following properties

(i) *It satisfies the Kolmogorov-Chapman equation*

$$
\int_{\mathbb{R}^d} G(t,x,z)G(s,z,y)\, dz = G(t+s,x,y), \quad t>0,\ s>0,\ x\in\mathbb{R}^d,\ y\in\mathbb{R}^d;
$$

(ii) *It is absolutely integrable and* $\int_{\mathbb{R}^d} G(t,x,y)\, dy \equiv 1$.

Proof. Formulas (4), (6), and (7) allows us to write down the inequality

$$
(9)\qquad |V_0(t,x,y)| \le \frac{N}{c\,\alpha} \frac{1}{(t^{1/\alpha}+|y-x|)^{d+\alpha-1}}
$$

Then the following inequality is true for all $k\in\mathbb{N}$ and $t>0$, $x\in\mathbb{R}^d$, $y\in\mathbb{R}^d$

$$
|V_k(t,x,y)| \le \frac{N}{c\alpha}\int_0^t d\tau \int_{\mathbb{R}^d} \frac{1}{((t-\tau)^{1/\alpha}+|z-x|)^{d+\alpha-1}} |V_{k-1}(\tau,z,y)||a(z)|\, dz
$$

Using inequality (7) one can show by induction on k that the function V_k for $k = 0, 1, 2, \dots$ satisfies the inequality

$$|V_k(t,x,y)| \le \|a\|_p^k \left(\frac{N}{c\,\alpha}\right)^{k+1} C^{k\nu} R_k \frac{t^{k\frac{\rho}{\alpha}}}{(t^{1/\alpha}+|y-x|)^{d+\alpha-1}} \le$$
$$\le \|a\|_p^k \left(\frac{N}{c\,\alpha}\right)^{k+1} C^{k\nu} R_k\, t^{(k\rho-d+1)\frac{1}{\alpha}-1},$$

where $\nu = 1-\dfrac{1}{p}$, $\rho = 1-\dfrac{d}{p}$, $R_0 = 1$, $R_k = R_{k-1}\left(B\left(\dfrac{p-d-\alpha}{\alpha(p-1)}, 1+(k-1)\dfrac{p-d}{\alpha(p-1)}\right) + +B\left(1, \dfrac{p-d-\alpha}{\alpha(p-1)} + (k-1)\dfrac{p-d}{\alpha(p-1)}\right)\right)^{1-\frac{1}{p}}$ (or limits of these expressions when p tends to infinity, if $p = +\infty$).

Therefore, the series on the right hand side of (5) converges uniformly in $x \in \mathbb{R}^d$, $y \in \mathbb{R}^d$ and locally uniformly in $t > 0$. Thus, the function V given by this equality is a solution of equation (3). In addition, the following inequality

$$|V(t,x,y)| \le C_T \frac{1}{(t^{1/\alpha}+|y-x|)^{d+\alpha-1}} \tag{10}$$

has been proved for $x \in \mathbb{R}^d$, $y \in \mathbb{R}^d$ and $0 < t \le T$, where C_T is a positive constant that, maybe, depends on $T > 0$.

Remark 2.1. The function $V(t,x,y)$ is the unique solution of equation (3) in the class of functions that satisfy inequality (10).

Finally, since the equality $(\mathbf{B}_x G(t,x,y), e(x)) = V(t,x,y)$ holds, the function

$$G(t,x,y) = g(t,x,y) + \int_0^t d\tau \int_{\mathbb{R}^d} g(t-\tau,x,z)V(\tau,z,y)|a(z)|\,dz, \tag{11}$$

is the perturbation of the transition probability density of the α-stable process.

Here we have used the following statement.

Lemma 2.1. *The equality* $\mathbf{B}_x \displaystyle\int_0^t d\tau \int_{\mathbb{R}^d} g(t-\tau,x,z)V(\tau,z,y)|a(z)|\,dz = = \int_0^t d\tau \int_{\mathbb{R}^d} \mathbf{B}_x g(t-\tau,x,z)V(\tau,z,y)|a(z)|\,dz$ *is true.*

The proof of this lemma is based on the following representation of the operator $\mathbf{B}$: $\mathbf{B}\varphi(x) = \dfrac{1}{\varkappa}\displaystyle\int_{\mathbb{R}^d} \frac{\varphi(x+y)-\varphi(x)}{|y|^{d+\alpha}}\,y\,dy$ for a bounded differentiable function $(\varphi(x))_{x\in\mathbb{R}^d}$, where $\varkappa = -\dfrac{2\pi^{\frac{d-1}{2}}\Gamma(2-\alpha)\Gamma\left(\frac{\alpha+1}{2}\right)\cos\frac{\pi\alpha}{2}}{(\alpha-1)\Gamma\left(\frac{d+\alpha}{2}\right)}$.

Proof. Let us consider a set of operators $\{\mathbf{B}^\varepsilon : \varepsilon > 0\}$ that act on a continuously differentiable bounded function $(\varphi(x))_{x\in\mathbb{R}^d}$ according to the following rule

$$\mathbf{B}^\varepsilon\varphi(x) = \frac{1}{\varkappa}\int_{|u|\ge\varepsilon} \frac{\varphi(x+u)-\varphi(x)}{|u|^{d+\alpha}}\,y\,dy.$$

It is clear that $\lim\limits_{\varepsilon\to 0+} \mathbf{B}^\varepsilon\varphi(x) = \mathbf{B}\varphi(x)$ for all $x \in \mathbb{R}^d$ and described above functions φ.

Inequalities (6) and (10) allow us to assert that

$$\left|\frac{u}{|u|^{d+\alpha}}(g(t-\tau,x+u,z)-g(t-\tau,x,z))V(\tau,z,y)|a(z)|\right| \le$$
$$\le \frac{const}{|u|^{d+\alpha-1}}\left(\frac{t-\tau}{((t-\tau)^{1/\alpha}+|z-x-u|)^{d+\alpha}}+\frac{t-\tau}{((t-\tau)^{1/\alpha}+|z-x|)^{d+\alpha}}\right)\times$$
$$\times\frac{1}{(\tau^{1/\alpha}+|y-z|)^{d+\alpha-1}}.$$

It is easy to see that the right hand side of this inequality is an integrable function with respect to (u,τ,z) on the set $\{|u|\ge\varepsilon\}\times(0;t)\times\mathbb{R}^d$ for all $t>0$ and $x\in\mathbb{R}^d$, $y\in\mathbb{R}^d$. Here we used formula (7). Therefore, we obtain the following equality

$$\mathbf{B}_x^\varepsilon\int_0^t d\tau\int_{\mathbb{R}^d} g(t-\tau,x,z)V(\tau,z,y)|a(z)|\,dz = \tag{12}$$
$$=\int_0^t d\tau\int_{\mathbb{R}^d}\mathbf{B}_x^\varepsilon g(t-\tau,x,z)V(\tau,z,y)|a(z)|\,dz,$$

using Fubini's theorem.

Inequalities (6), (7) and $|\mathbf{B}_x g(t,x,y)|\le\dfrac{const}{(t^{1/\alpha}+|y-x|)^{d+\alpha-1}}$ allow us to assert that the integral $\displaystyle\int_0^t d\tau\int_{\mathbb{R}^d}\mathbf{B}_x g(t-\tau,x,z)V(\tau,z,y)|a(z)|\,dz$ exists. Now we have to pass to the limit as $\varepsilon\to 0+$ in equality (12) to complete the proof of lemma. □

Let us prove that the function $G(t,x,y)$ satisfies the Kolmogorov-Chapman equation

$$G(t+s,x,y)=\int_{\mathbb{R}^d} G(s,x,z)G(t,z,y)\,dz \tag{13}$$

for all $s>0$, $t>0$, $x\in\mathbb{R}^d$, $y\in\mathbb{R}^d$. Note, that the function $g(t,x,y)$ satisfies equation (13).

Put $U(s,x,\varphi)=\displaystyle\int_{\mathbb{R}^d} G(s,x,y)\varphi(y)\,dy$, $u(s,x,\varphi)=\displaystyle\int_{\mathbb{R}^d} g(s,x,y)\varphi(y)\,dy$, and $W(s,x,\varphi)=\displaystyle\int_{\mathbb{R}^d} V(s,x,y)\varphi(y)\,dy$, where $\varphi\in C_b(\mathbb{R}^d)$.

Note, that the function $W(t,x,\varphi)$ is the unique solution of the following equation

$$W(t,x,\varphi)=W_0(t,x,\varphi)+\int_0^t d\tau\int_{\mathbb{R}^d} V_0(t-\tau,x,z)W(\tau,z,\varphi)|a(z)|\,dz, \tag{14}$$

where $W_0(s,x,\varphi)=\displaystyle\int_{\mathbb{R}^d} V_0(s,x,y)\varphi(y)\,dy$.

Then the function $U(s,x,\varphi)$ can be given by the equality (see (11))

$$U(t,x,\varphi)=u(t,x,\varphi)+\int_0^t d\tau\int_{\mathbb{R}^d} g(t-\tau,x,z)W(\tau,z,\varphi)|a(z)|\,dz$$

Now, let us find the function $U(t+s,x,\varphi)$. We have

$$U(t+s,x,\varphi)=u(t+s,x,\varphi)+\int_0^{t+s} d\tau\int_{\mathbb{R}^d} g(t+s-\tau,x,z)W(\tau,z,\varphi)|a(z)|\,dz=$$
$$=\int_{\mathbb{R}^d} g(s,x,y)u(t,y,\varphi)\,dy+\int_{\mathbb{R}^d} g(s,x,y)\,dy\int_0^t d\tau\int_{\mathbb{R}^d} g(t-\tau,y,z)W(\tau,z,\varphi)|a(z)|\,dz+$$
$$+\int_t^{s+t} d\tau\int_{\mathbb{R}^d} g(t+s-\tau,x,z)W(\tau,z,\varphi)|a(z)|\,dz=$$
$$=\int_{\mathbb{R}^d} g(s,x,y)U(t,y,\varphi)\,dy+\int_0^s d\tau\int_{\mathbb{R}^d} g(s-\tau,x,z)W(t+\tau,z,\varphi)|a(z)|\,dz.$$

Therefore, the function $W_t(s,x,\varphi)=W(t+s,x,\varphi)$ satisfies equation (14), where the function φ is replaced by $U(t,\cdot,\varphi)$. Then $W(t+s,x,\varphi)=W(s,x,U(t,\cdot,\varphi))$ and we arrive at the equality $U(t+s,x,\varphi)=U(s,x,U(t,\cdot,\varphi))$ or, what is the same,

$$\int_{\mathbb{R}^d} G(t+s,x,y)\varphi(y)\,dy=\int_{\mathbb{R}^d} G(s,x,z)\int_{\mathbb{R}^d} G(t,z,y)\varphi(y)\,dy\,dz= \\ =\int_{\mathbb{R}^d}\varphi(y)\,dy\int_{\mathbb{R}^d} G(s,x,z)G(t,z,y)\,dz.$$

Then relation (13) is proved because the function φ is an arbitrary bounded continuous one.

Next, we get $\int_{\mathbb{R}^d} G(t,x,y)\,dy\equiv 1$ from (2) and (3), because the equalities

$$\int_{\mathbb{R}^d} g(t,x,y)\,dy=1 \quad\text{and}\quad \int_{\mathbb{R}^d} V_0(t,x,y)\,dy=\left(\mathbf{B}_x\int_{\mathbb{R}^d} g(t,x,y)\,dy, e(x)\right)=0$$

for all $t>0$, $x\in\mathbb{R}^d$ are obvious, and the uniqueness of the solution of equation (3) leads us to the identity $\int_{\mathbb{R}^d} V(t,x,y)\,dy\equiv 0$. □

Remark 2.2. The family of operators $(T_t)_{t>0}$ defined for any bounded continuous function φ on $\mathbb{R}^d$ by the equality $T_t\varphi(x)=\int_{\mathbb{R}^d} G(t,x,y)\varphi(y)\,dy,\quad t>0,\ x\in\mathbb{R}^d$, indeed constitutes a semigroup generated by the operator $\mathbf{A}+(a(x),\mathbf{B})$. But, there is no Markov process in $\mathbb{R}^d$ corresponding to this semigroup because it does not preserve the property of a function to take on only non-negative values (see, for example, [1]).

3. The Cauchy problem

First, let the function a be smooth enough. For the simplicity we suppose that $a\in C_0^\infty(\mathbb{R}^d)$ (this is the space of all $\mathbb{R}^d$-valued infinitely differentiable functions on $\mathbb{R}^d$ with compact support). Thus, the function

$$U(t,x)=\int_{\mathbb{R}^d}\varphi(y)G(t,x,y)\,dy= \\ =\int_{\mathbb{R}^d}\varphi(y)g(t,x,y)\,dy+\int_0^t d\tau\int_{\mathbb{R}^d} g(t-\tau,x,y)\int_{\mathbb{R}^d} V(\tau,y,z)\varphi(z)\,dz|a(y)|\,dy$$

is the unique (in the class of functions that tends to zero at infinity) solution of the Cauchy problem (8) with $f(t,x)=|a(x)|\int_{\mathbb{R}^d} V(t,x,z)\varphi(z)\,dz$.

Now we note that $V(t,x,y)=(\mathbf{B}_xG(t,x,y),e(x))$. Then

$$f(t,x)=\int_{\mathbb{R}^d}(\mathbf{B}_xG(t,x,z),a(x))\varphi(z)\,dz=(a(x),\mathbf{B}_xU(t,x))$$

and the function $U(t,x)$ is a solution of the Cauchy problem

$$\begin{cases} \frac{\partial u(t,x)}{\partial t}=\mathbf{A}_xu(t,x)+(a(x),\mathbf{B}_xu(t,x)), & t>0,\ x\in\mathbb{R}^d,\\ \lim\limits_{t\to 0+}u(t,x)=\varphi(x), & x\in\mathbb{R}^d \end{cases} \tag{15}$$

for an arbitrary continuous bounded function $(\varphi(x))_{x\in\mathbb{R}^d}$.

The next statement will allow us to construct a generalized solution of the Cauchy problem.

Theorem 3.1. *Let a and $\tilde{a}$ be given functions that satisfy the conditions of Theorem 2.1. Denote by G and $\tilde{G}$ the solutions of* (1) *corresponding to the functions a and $\tilde{a}$,* *respectively. Then the inequality*

$$|G(t,x,y)-\tilde G(t,x,y)| \le H_T\|a-\tilde a\|_p \frac{t^{1-\frac{d}{\alpha p}}}{(t^{\frac{1}{\alpha}}+|y-x|)^{d+\alpha-1}}$$
$$(\textit{or } |G(t,x,y)-\tilde G(t,x,y)| \le H_T\|a-\tilde a\|_\infty \frac{t}{(t^{\frac{1}{\alpha}}+|y-x|)^{d+\alpha-1}},\ \textit{if } p=+\infty)$$

is held on each domain $(0,T]\times\mathbb{R}^d\times\mathbb{R}^d$ *for* $T<+\infty$*, where the positive constant* H_T *depends on* c*,* α*,* $\|a\|_p$*,* $\|\tilde a\|_p$ *and* T*.*

Proof. We will consider the case of finite values of p. The case $p=+\infty$ is similar to this one.

It is easy to see that

$$G(t,x,y)-\tilde G(t,x,y) = \int_0^t d\tau \int_{\mathbb{R}^d} g(t-\tau,x,z)W(\tau,z,y)\,dz, \tag{16}$$

where $W(\tau,z,y) = V(\tau,z,y)|a(z)| - \tilde V(\tau,z,y)|\tilde a(z)|$ and the functions V and $\tilde V$ are solutions of equation (3) with the functions a and $\tilde a$, respectively. We can write down the following equality

$$\begin{aligned} W(t,x,y) = W_0(t,x,y) + |a(x)|\int_0^t d\tau \int_{\mathbb{R}^d} V_0(t-\tau,x,z)W(\tau,z,y)\,dz+\\ +\int_0^t d\tau\int_{\mathbb{R}^d} W_0(t-\tau,x,z)\tilde V(\tau,z,y)|\tilde a(z)|\,dz, \end{aligned} \tag{17}$$

taking into account equality (3), where $W_0(t,x,y) = (\mathbf{B}_x g(t,x,y), a(x)-\tilde a(x))$.

Let us estimate the first and the third items on the right-hand side of equality (17). The following inequality

$$|W_0(t,x,y)| \le |\mathbf{B}_x g(t,x,y)||a(x)-\tilde a(x)| \le \frac{N}{c\alpha}\frac{|a(x)-\tilde a(x)|}{(t^{1/\alpha}+|y-x|)^{d+\alpha-1}}$$

is easily derived from formulas (4) and (9) for $x\in\mathbb{R}^d$, $y\in\mathbb{R}^d$, $t>0$. Using inequalities (7), (10) and the previous inequality one can show that for $x\in\mathbb{R}^d$, $y\in\mathbb{R}^d$, $t\in(0,T]$ and every $T>0$

$$\left|\int_0^t d\tau\int_{\mathbb{R}^d} W_0(t-\tau,x,z)\tilde V(\tau,z,y)|\tilde a(z)|\,dz\right| \le K_T|a(x)-\tilde a(x)|\frac{t^{1/\alpha}}{(t^{1/\alpha}+|y-x|)^{d+\alpha-1}},$$

where K_T is some positive constant, which depends on T, maybe.

Thus, we can write down the following inequality

$$\begin{aligned} |W(t,x,y)| \le Q_T\frac{|a(x)-\tilde a(x)|}{(t^{1/\alpha}+|y-x|)^{d+\alpha-1}}+\\ +\frac{N}{c\alpha}\int_0^t d\tau\int_{\mathbb{R}^d}\frac{|W(\tau,z,y)|}{((t-\tau)^{1/\alpha}+|z-x|)^{d+\alpha-1}}\,dz \end{aligned} \tag{18}$$

that holds true for $x\in\mathbb{R}^d$, $y\in\mathbb{R}^d$, $t\in(0,T]$ and every $T>0$, where $Q_T>0$ is some constant, which maybe depends on T.

Iterating inequality (18) we obtain for $x\in\mathbb{R}^d$, $y\in\mathbb{R}^d$, $t\in(0,T]$ and every $T>0$

$$|W(t,x,y)| \le \sum_{k=0}^{\infty} R_k(t,x,y), \tag{19}$$

where $R_0(t,x,y) = Q_T\frac{|a(x)-\tilde a(x)|}{(t^{1/\alpha}+|y-x|)^{d+\alpha-1}}$ and for $k\ge1$ the following recurrence relation $R_k(t,x,y) = \frac{N}{c\alpha}\int_0^t d\tau\int_{\mathbb{R}^d}\frac{R_{k-1}(\tau,z,y)}{((t-\tau)^{1/\alpha}+|z-x|)^{d+\alpha-1}}\,dz$ holds.

Using Hölder's inequality and inequality (7) one can show by induction on k that the function R_k for $k = 1, 2, \dots$ satisfies the inequalities

$$0 \le R_k(t,x,y) \le Q_T \left(\frac{N}{c\alpha}\right)^k C^{k-1/p} \left(2B\left(1, \frac{p-d-\alpha}{\alpha(p-1)}\right)\right)^{1-1/p} \times$$
$$\times \left(B\left(\frac{1}{\alpha}, 1+\frac{p-d}{\alpha p}\right) + B\left(1, 1+\frac{2p-d}{\alpha p}\right)\right) \times \dots$$
$$\times \left(B\left(\frac{1}{\alpha}, 1+\frac{(k-1)p-d}{\alpha p}\right) + B\left(1, 1+\frac{kp-d}{\alpha p}\right)\right) \times$$
$$\times \frac{t^{(kp-d)/(\alpha p)}}{(t^{1/\alpha}+|y-x|)^{d+\alpha-1}} \|a-\tilde{a}\|_p.$$

Hence, we conclude that the series in inequality (19) converges uniformly in $x \in \mathbb{R}^d$, $y \in \mathbb{R}^d$ and locally uniformly in $t > 0$. Therefore, the following inequality

$$|W(t,x,y)| \le M_T \frac{\|a-\tilde{a}\|_p}{(t^{1/\alpha}+|y-x|)^{d+\alpha-1}} t^{\frac{p-d}{\alpha p}} + Q_T \frac{|a(x)-\tilde{a}(x)|}{(t^{1/\alpha}+|y-x|)^{d+\alpha-1}}$$

holds for $x \in \mathbb{R}^d$, $y \in \mathbb{R}^d$, $t \in (0,T]$ and every $T > 0$, where M_T and Q_T are some positive constants, which maybe depend on T.

Some not difficult calculations using formulas (6),(7), (16) and Hölder's inequality lead us to the assertion of the theorem. □

Corollary 3.1. *Let $\varphi \in C_b(\mathbb{R}^d)$ and G, $\tilde{G}$ be as in Theorem 2.1. Put*

$$U(t,x) = \int_{\mathbb{R}^d} G(t,x,y)\varphi(y)\,dy, \quad \tilde{U}(t,x) = \int_{\mathbb{R}^d} \tilde{G}(t,x,y)\varphi(y)\,dy.$$

Then the following inequality $|U(t,x) - \tilde{U}(t,x)| \le L_T \sup_y |\varphi(y)| \|a-\tilde{a}\|_p$ is held for $x \in \mathbb{R}^d$, $0 < t \le T$. Here L_T is some positive constant, that maybe depends of T.

Now, let $a(x)$ be a given $\mathbb{R}^d$-valued function on $\mathbb{R}^d$ satisfying the condition $\|a\|_p < \infty$ for some $p > d + \alpha$. Then there exists a sequence of functions $a_n \in C_0^\infty(\mathbb{R}^d)$, such that $\|a_n - a\|_p \to 0$ as $n \to \infty$. According to Corollary 3.1, we can defined the function $U(t,x)$ by the equality $U(t,x) = \lim_{n\to\infty} U_n(t,x)$, where $U_n(t,x)$ is the solution of the Cauchy problem (15) corresponding to the function a_n. The statement of Theorem 3.1 means that $U(t,x) = \int_{\mathbb{R}^d} G(t,x,y)\varphi(y)\,dy$, where $G(t,x,y)$ is the perturbation (corresponding to the function a) of the transition probability density of the symmetric stable process (see Definition 1.1). We say exactly in this sense that the function $U(t,x)$ is a generalized solution of the Cauchy problem (15).

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